JANET BOWERS and HELEN M. DOERR

AN ANALYSIS OF PROSPECTIVE TEACHERS' DUAL ROLES IN UNDERSTANDING THE MATHEMATICS OF CHANGE: ELICITING GROWTH WITH TECHNOLOGY[1]

ABSTRACT. We analyze the interrelations between prospective and practicing teachers' learning of the mathematics of change and the development of their emerging understanding of effective mathematics teaching. The participants in our study, who were all interested in teaching secondary mathematics, were mathematics majors who had significant formal knowledge of the fundamental concepts of calculus prior to taking our courses, but who often experienced and expressed procedural orientations toward the teaching of mathematics. To address this difficulty, we developed novel computer-based activities to challenge the participants' mathematical understandings and required them to use technology during short teaching episodes they conducted with younger students. To analyze our participants' understandings, we developed a framework that juxtaposes the roles of the participants as students and teachers, and their understanding of mathematics and of pedagogical strategies. Our analysis of the participants' views from these different perspectives enabled us to see simultaneously the intertwined development of subject matter insights and specific views of teaching.

Students enrolled in mathematics education courses are simultaneously learners and teachers in transition. As learners, they are constructing new ways of thinking about seemingly familiar mathematics and about new ways that others might learn. As teachers in transition, they are anticipating how their experiences in learning mathematics will relate to their future experiences as teachers in their own classrooms. There are several difficulties facing mathematics educators teaching such courses for preservice teachers. One obstacle relates to prospective teachers' well-documented resiliency toward changing their views of effective pedagogy (cf., Cooney, Wilson, Albright & Chauvot, 1998; Hiebert, 1986; Lampert & Ball, 1998; Pajares & Bengston, 1995; Siebert, Lobato & Brown, 1998; Thompson, 1992). A second difficulty is that, as learners, prospective teachers are often content with what may be superficial understandings of deep mathematical concepts. Once they have formalized a procedure, it is difficult to re-visit the underlying concept for deeper understanding (Hiebert & Carpenter, 1992; Lee & Wheeler, 1989; Skemp, 1978; Wilson & Goldenberg, 1998). As teacher educators, we would like pre-service teachers to realize that fragile mathematical understandings are inade-

Journal of Mathematics Teacher Education **4:** 115–137, 2001.

quate when teaching mathematics in ways that support more meaningful understanding.

Our approach to challenging and perhaps changing prospective teachers' understandings of mathematics and effective pedagogy was to design novel computer-based activities that would elicit dissonance between what was expected and what occurred on the computer screen, and then to discuss how this dissonance could promote reflection and learning. The goal of our study was to investigate two questions:

1) How do prospective teachers, acting in the role of *students*, think about the mathematics of change when using an exploratory microworld in the context of their course work?
2) How do prospective teachers, acting in the role of *teachers*, think about the mathematics of change when using an exploratory microworld during tutoring sessions with young children?

THEORETICAL FRAMEWORK

The theoretical framework that guides our work is based on a constructivist perspective in which learning is viewed as a process of experiencing dissonance and working to resolve perturbations by building viable explanations (von Glasersfeld, 1987, 1995). As Steffe and Thompson (2000) recently noted, Piaget contended that there are four factors that contribute to one's cognitive development. These include social interaction, maturation, physical experience, and self-regulation. "Individuals establish equilibrium among personal schemes of action and anticipation as they interact in mutual adaptation – as constrained by local limitations imposed by their abilities to accommodate those very schemes" (Steffe & Thompson, 2000, p. 193). The critical element of this general model is that students are seen as cognizing individuals who are continually interacting with each other and with their environment (which includes the computer and the accompanying activities) and adapting their own views through processes of interactive accommodation. We take very seriously Steffe and Thompson's recommendation that "Researchers should not apply general models like von Glasersfeld's or Vygotsky's directly to the practice of mathematics education" (p. 204). In fact, we view the model as a general way of looking at how the participants in our study accommodated their current ways of knowing mathematics with the unanticipated outcomes they experienced during some of their activities. To create the need for our participants to adapt their mathematical understandings and their views of effective pedagogy, we designed activities that were grounded in the well-established body of research regarding students' conceptions of the mathematics of change. The nexus of this research consists of studies describing students' understandings of motion and graphing (Bowers & Nickerson, in press; Cooney & Wilson, 1993; Kaput & Roschelle, 1997; Nemirovsky & Monk, 2000), of rate (Harel, Behr, Lesh & Post, 1994; Lobato & Thanheiser, 1999; Thompson & Thompson, 1996; Thompson, 1994, 1996), and of calculus (Davis & Vinner, 1986; Kaput, 1994; Lauten, Graham & Ferrini-Mundy, 1994; Schoenfeld, Dubinsky & Gleason, 1997; Tall, 1992; Thompson, 1994; Williams, 1991).

One consistent finding in the research regarding the mathematics of change is the difficulty students have reading and interpreting graphs of motion. For example, several researchers have identified the tendency for students to interpret the graph of position versus time as a picture of the actual path of the motion (cf., Leinhardt, Zaslavsky & Stein, 1990). This tendency towards *iconic translation* of the graph as a picture of the physical event suggests that graphs of motion and the change in motion are difficult for students to construct and interpret. Likewise, students encounter difficulties in interpreting the global features of a graph, such as change over time (Monk, 1992). Thus, we anticipated that *graph as path* and point-wise versus over-time graphical interpretations might serve as potential sources of perturbation in the computer-based activities.

Other studies have revealed that students have particular difficulty understanding graphs of rates, since they do not have strong intuitions about rate prior to instruction (cf., Harel, Behr, Lesh & Post, 1994). In the case of the mathematics of change, one possible reason that rate of change (speed or velocity) is not as intuitive as position is that speed is an intensive quantity whereas position is an extensive (measurable) quantity (Schwartz, 1988). In exploring the distinction between intensive and extensive quantities, Lobato and Thanheiser (1999) found that students' everyday experiences with an intensive quantity like speed did not help them form meaningful ratios for measuring the speed of novel motions such as a mouse running along the floor. In fact, in some cases students' prior experiences may have promoted their tendencies to conflate the various quantities that could be measured in the situation. These findings guided our efforts to design tasks that would involve participants in activities that demanded strong proportional reasoning about rates and their relation to accumulated position in order to interpret the relationships between the quantities in the computer-generated graphs.

Given that the mathematics of change is a central foundation for calculus, we also drew on research on students' difficulties in developing conceptual understandings of limits and functions. Although all participants had taken at least three undergraduate courses in calculus, we did not assume that their understanding and experiences extended beyond superficial interpretations of differentiation and integration (Selden, Mason, & Selden, 1989; Tall, 1992; Thompson, 1994). Thompson (1994) explained that the danger of such fragile knowledge is that students who develop procedural understanding often think of algebraic expressions as commands to do something, i.e., calculate, rather than as quantities that are mathematical objects in and of themselves. For example, students may view the expression $7x - x^2$ as a string of operations rather than as an entity in itself, namely a function of x. One implication of this view is that students come to view the integral of an algebraic expression as simply a formalized algorithm; they do not view the expression as a position function that represents the accrual of distance resulting from traveling at a given velocity over time. We hypothesized that if such procedural understanding was a prominent aspect of our own students' understanding, then asking them to create position graphs over time based on velocity graphs (rather than on their algebraic representations) would be a potential source for perturbation. Our hypothesis was that, for them, position graphs were the result of algebraic manipulations, not graphical interpretations. Likewise, being able to interpret families of functions and the effects of parameters, such as adding the constant C when computing an integral, would be potentially challenging in that such interpretation depends on an understanding of the integral as a family of position functions all determined by the same velocity function.

In summary, the constructivist learning theory that guided our work was based on efforts to initiate perturbations in our participants' views of the mathematics of change and elicit shifts in their views of effective ways of teaching these concepts. The way in which we initiated dissonance was to create novel, computer-based activities that might challenge our students' expectations. In creating these activities we drew on three main areas of research: (a) students' propensity to view a graph as a picture of the path traveled; (b) students' difficulties in interpreting intensive quantities and their graphs; and (c) the difficulties inherent in superficial understandings of calculus and the mathematics of change.

METHODS

Participants and Setting

The participants for this study were pre- and in-service secondary mathematics teachers enrolled in one of two technology-based mathematics courses taught by each of the authors. The courses were taught at two different universities on opposite coasts of the United States. The goal of both courses was to expand students' ideas about the mathematics of change in conceptual ways by engaging the participants in three shared instructional sequences. The sites differed slightly in terms of course emphasis and participants' educational levels and experiences.

The course at Site A had 15 students, five of whom were in-service teachers enrolled in a master's degree program. The remaining ten students were undergraduate mathematics majors, most of whom were seniors, who had volunteered as teaching assistants in local schools, but had not yet been certified to teach. The course at Site B had eleven students; ten were preservice master's or doctoral students, and one was an in-service master's student. All the participants at Site B had completed student teaching at the secondary level, and most had taught introductory level courses as teaching assistants at the university level. All students enrolled in the courses had completed all or most of the courses required for an undergraduate mathematics major. Each course met for 3 hours per week with some computer lab work completed in class and the remainder completed by the students as homework. All students enrolled in the two courses agreed to participate in the study.

Technology Environment

The shared activity sequences enacted at each site involved the use of motion detectors and the MathWorlds software environment, a simulation world developed by a team of researchers at the University of Massachusetts at Dartmouth (Kaput & Roschelle, 1997).[2] This environment is a dynamic microworld for exploring one-dimensional motion in which any combination of three graphs (position vs. time, velocity vs. time, and acceleration vs. time) can be linked to an animated simulation and to each other. Unlike most function graphing software that includes multiple, linked representations of the same data set, the central focus of the MathWorlds software is the exploration of the same phenomena that can be represented with different data sets (i.e., position, velocity, and acceleration). In other words, unlike function graphing software that includes linked tables of values, graphs, and algebraic equations to represent the same data set, the MathWorlds software includes an animation of a character or a set of characters moving in a horizontal direction that is directly linked to its position, velocity and acceleration graphs, which also are bi-directionally linked to each other. This bi-directional link enables learners to change any parameter of a character's motion by manipulating any of the graphical representations.

Instructional Sequences

The three core instructional sequences that were shared between the two courses were designed to engage participants in experiential and graphical ways of challenging their formal knowledge of the mathematics of change and to support the development of pedagogical content knowledge regarding how younger learners might engage with these ideas. The first two sequences, which involved investigations of relative and parabolic motions, were specifically designed to provide opportunities to explore the richness of the Fundamental Theorem of Calculus by examining the relationship between a velocity graph and its linked position graph. The third activity sequence involved having the participants design, implement, and reflect on a MathWorlds-based lesson sequence to help younger students interpret various concepts of the mathematics of change.

Sequence 1. The core idea of the first sequence was to create a situation in which the relative, one-dimensional motion of two characters could be investigated. In each task, one character travels at a constant rate while another travels at a linearly increasing or decreasing rate. A series of questions focused on determining when and if the two characters will meet (e.g., whether or not a Cheetah would catch a Gazelle) under a variety of conditions. The level of difficulty increased from simple chases in which the two characters started at the same time to more complex scenarios in which one character got a head-start in time (an x-axis translation) or a head-start in distance (a y-axis translation on the position graph only). When investigating overtake questions, the students were only given information about each animal's velocity and were asked to solve the task by creating a velocity graph and a linked simulation as shown in Figure 1.

The pivotal aspect of this sequence was that the position graph of each animal's motion was never used. We attempted to perturb the students' thinking about their views of the relation between position and velocity graphs by asking them to focus on how they could determine a final position by thinking primarily in terms of the graphical representation of the relative velocities. In this way, their prior knowledge of computing integrals algorithmically would not support their efforts to interpret final position because they were not given any algebraic expressions. We instead anticipated that they would reorganize their views of velocity graphs as showing an accumulation of distance based on rate traveled up to the current moment of time.

Sequence 2. The second investigation involved the use of a motion detector to graph the position and velocity of a bouncing ball and then to simulate

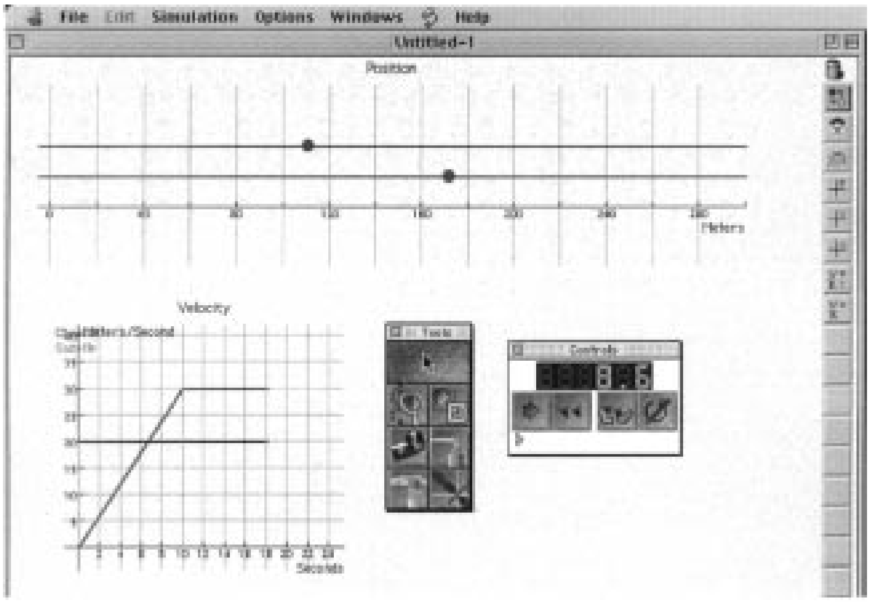


Figure 1. Velocity graph (left) and Dots simulation (right) used with Cheetah and Gazelle activities.

this experience in MathWorlds. We designed this activity to accentuate the contrast between using the motion detector, which takes the motion of a ball as input and gives graphical data as the *output*, and the MathWorlds software, which essentially inverts this process by taking idealized graphical data as the *input* and giving an animation of a bouncing ball phenomena as output. As with the first sequence, we intended that a pedagogical inversion, that is, a reversal of the traditional instructional approach, would encourage the participants to act in new ways with familiar mathematical objects and re-think their understanding of the relationship between rates and accumulations.

Sequence 3. The third shared activity sequence was designed to engage the participants in their roles as teachers. Each participant was asked to create a three-lesson sequence that focused on the mathematics of change and utilized the MathWorlds software. The participants were allowed to choose the grade level (i.e., middle or high school), the content as long as it involved a concept within the mathematics of change, and the pedagogical approach, such as working with an individual student or a small group, or using a single context or multiple contexts for developing concepts. After each teaching session, the participants reflected on the lesson and modified subsequent lessons. The intent of this activity was to have the participants reflect on and consequently modify their view of their actions within the microworld as they shifted from the role of student to the role of teacher.

Data and Analytic Method

The data consisted of copies of the participants' written work on the relative motion assignment and the Bouncing Ball assignment; written reflections on their teaching; and the instructors' daily teaching journals. Data analysis involved comparison of the data from the two courses in three phases. In the first phase, each author identified the most striking trends in her students' mathematical and pedagogical thinking by analyzing all student work and reviewing journal notes from each day's instruction. A trend was defined as an observed reorganization in thinking or an "Aha" insight reported by a majority of students over the course of an activity sequence. The second phase involved having each instructor compare, contrast, and elaborate of each of the trends in order to differentiate constructs that could be linked, at least in part, to the students' participation in the common activities from those that were more likely specific to the norms and values developed at a particular site. The third phase of the analysis involved the documentation of the occurrences of the final list of the trends in the data from each course. Thus, all of the trends reported in this paper were observed at both sites.

RESULTS

Our primary goal was to identify and understand the sources and types of change that we observed in participants' views of pedagogy and their understandings of the mathematics of change. We found it useful to categorize the different types of trends the participants reported as they assumed the dual roles of student and teacher. In Figure 2, Cells I and IV refer to the more familiar paradigm in which mathematical knowing is examined from the perspective of participants as students (Cell I), and pedagogical knowing is examined from the perspective of participants as teachers (Cell IV). Our hypothesis, however, was that the experiences and insights listed in Cells II and III also contributed to the participants' overall changes in their mathematical and pedagogical knowledge. In the following discussion of our results, we describe our findings in each of these four analytic categories in more detail.

	Mathematical Knowing	Pedagogical Knowing
Participants as Students	I • Realizing that a velocity graph determines a family of position graphs • Reconceptualizing the Mean Value Theorem	II • Debating the value of conceptual explanations • Discussing the explore vs. symbolize dichotomy
Participants as Teachers in Transition	III • Differentiating local and global interpretations of graphs • Viewing the "race" as a powerful metaphor	IV • Recognizing the value of building on students' incorrect explanations • Recognizing the supports and constraints of the technology

Figure 2. Participants' mathematical and pedagogical insights when acting with the Mathworlds software.

Mathematical Insights from Participants as Students

In this section, we describe two mathematical insights that our participants reported after experiencing perturbations in their work as mathematics students using the MathWorlds software at each of the two sites.

Mathematical insight #1: A velocity graph determines a family of position graphs. One insight that all participants from both sites reported was that any given velocity graph determines a family of position graphs. The genesis of this insight occurred as follows. First, participants translated the velocity graph vertically and noted that the linked position graph moved accordingly. Next, they translated the position curve vertically, but noted that the linked velocity curve did not change at all. Their efforts to resolve this perturbation ultimately led them to develop a deeper understanding between the underlying quantities represented in velocity and position graphs. They came to see that in varying the initial starting point by vertically translating the position graph, they were generating a family of position graphs but were not changing the speed at which the animal moved. At this point, many students reported an "Aha" insight regarding the meaning of the ubiquitous "+*C*" they had routinely been adding when computing indefinite integrals in their prior calculus classes. One participant from Site A wrote:

First I made a velocity graph for the Gazelle, and then transferred this to a position graph. Then by chance I moved the final position of the Gazelle's dot [in the simulation] and the Gazelle's line on the position graph shifted upward! "Great," I thought, I messed everything up, now I'll have to redo the graphs – that is when it hit me. The velocity graph had not changed. The velocity graph reflects the rates on the position graph – not the starting point. With a different starting point (simply shifted all up) all of the velocities remain the same.

> Cool! I guess I knew that, but now I am aware of it and understand the workings behind it. . . . I did not ever realize (probably because it was never posed to me) that from the velocity graph you could not draw the position graph, unless you were given the starting point!

This participant described her surprise when she realized that what she expected would happen, that is, that moving the character in the simulation would mess up the velocity graph, did not occur. She resolved her perturbation by forming what, to her, was a more viable interpretation of the velocity graph: "the velocity graph reflects the rates on the position graph – not the starting point." This reconsideration of the velocity graph brought forth another aspect of her previous formal knowledge, namely the value of the *y*-intercept as a starting point. She also noted that she probably already was aware of this, but had not fully realized "the workings behind it."

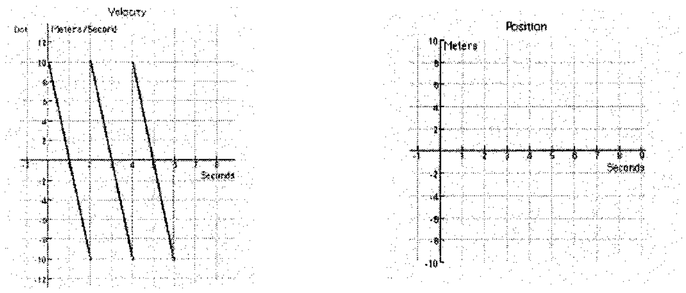
Mathematical insight #2: The difference between average and instantaneous velocity. The Bouncing Ball activity required the participants to create a position graph given the velocity graph shown in Figure 3a. Like the first sequence, we anticipated that this task would be difficult for students with fragile understandings of calculus who relied on rote methods of integration and who had not formed an image of position in terms of *accumulated distance* accrued by traveling at a linearly changing rate over time. Our goal was to challenge the participants' current mathematical knowledge as well as their views about the traditional ways the subject often is taught. As we anticipated, when first attempting this task with paper and pencil, over half of the participants at Site A attempted to solve the task by relying on the formula of d = r * t. Thus, these participants created a table of values with three columns as shown in Figure 3b, and then calculated the position at any given time *x* by multiplying the velocity at that point (as indicated by the *y*-coordinate of the velocity graph at time *x*) and the value of the time at that point. One participant described his work by noting, "In order to obtain the position graph, you must multiply the velocity by the time to have the desired units of meters. The algorithm looks like this, (m/s) * (s) = m." This student's position vs. time graph is shown in Figure 3c.

When this participant compared the graph he had created on paper with the position graph that he created in the MathWorld, he was surprised to see that they did not match. This observation, which he brought up in class, led to a discussion of two critical mathematical concepts: the need for the conventional definition of negative velocity and the difference between average and instantaneous speed. These discussions led the class to devise a more meaningful interpretation of the Mean Value Theorem based on a graphical interpretation of rate. The average rate was defined as

(a)

Questions:

1) Consider the following velocity graph. What would a graph of position v. time look like?



(b)

	A	B	C
1	**Time**	**Velocity**	**Position**
2	0	10	0
3	0.5	5	2.5
4	1	0	0
5	1.5	-5	-7.5
6	1.75	-8	-14
7	2	0	0
8	2.1	9	18.9
9	2.5	5	12.5
10	3	0	0

(c)

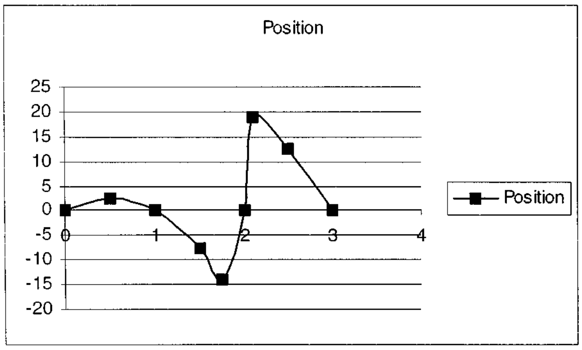


Figure 3. (a) Assignment asking students to create a graph that shows the ball's position at any time given the velocity graph (assume $P(0) = 0$). (b) One student's solution to assignment. (c) Student's graph created by plotting time and product of $t * v(t)$.

the constant rate at which another character would travel in order to cover the same distance as the bouncing ball during the same given time interval.

Pedagogical Insights from Participants as Students

In this section, we continue to view the participants as students learning mathematics but now consider the ways in which their pedagogical thinking was challenged as they worked with the microworld activities.

Pedagogical insight #1: The potential value of conceptual explanations. At both sites, the focus of most activities included an emphasis on explaining why a computer-generated graph appeared as it did. At Site A, this was discussed in terms of a distinction between calculational and conceptual explanations. Although the instructors had hoped that such a practice would be helpful, some of the participants had difficulties understanding the purpose and form that such explanations should take. Several of the participants questioned the value of this practice whereas others maintained that conceptual explanations supported their own efforts to develop imagery for motion and hence would support their future students' efforts to reason conceptually as well. Although the class remained undecided, they did agree that teaching with technology involves rethinking the format of activities and what counts as an acceptable explanation and solution within any given classroom culture.

Pedagogical insight #2: The tradeoffs in having exploration before or after symbolization. Participants discussed extensively whether mathematical formalisms should be introduced before, during, or after students have explored the mathematics that the symbols portend to signify (cf., Doerr, 1997). Given that all of the participants in the study had already encountered the formal symbols of calculus, and hence the formalisms of calculus preceded their explorations in graphically oriented microworld, one might expect that they would argue for the symbolize-then-explore approach. Indeed, some of the participants at each site maintained that this was a desirable pedagogical ordering given that it served as a source of perturbation and ultimate reorganization for them. On the other hand, others were eager to shift their pedagogical approach, based in part on their enthusiasm for their own new-found conceptual insights and on the potential for learners to meaningfully engage in conceptually oriented activities. The point here is not that the students should have come to an agreement, or even that there is one right answer to the question. Instead, the value of these discussions was that the participants assumed the roles of both teacher and student as they argued their points. Moreover, they realized that they, as prospective teachers, do have choices in how they interact with their students and that these choices affect students' views of mathematics in general.

Mathematical Insights from Participants as Teachers

In this section, we shift from viewing our participants as students of mathematics to viewing them as teachers who were tutoring and teaching younger children in one-on-one or small-group settings with the MathWorlds software. Given that this activity occurred toward the end of the semester at both course sites, we expected to see some evidence of the participants' increased pedagogical content knowledge based on our curricular agendas as described earlier.

Across both sites, 38% of the participants chose to focus their lesson on the relationship between the position and velocity graphs and, in particular, the concept relating area under the velocity graph accrued at each time x to the value of the object's position at time x. This was not surprising, because the two primary activity sequences in the course (the Cheetah and Gazelle and the Bouncing Ball activities) focused on aspects of this concept. As a consequence of their experienced instruction, this may have appeared to the participants as a natural starting point for their activities with younger learners. What was more surprising was that the remaining 62% of the participants extended their own learning experiences with MathWorlds by designing learning activities for students that addressed other mathematical content (such as y-intercept and slope) in novel ways.

In the following two sections, we describe the mathematical and pedagogical insights that emerged as the participants transitioned from their role as students to their role as teachers. First, we present two mathematical insights that emerged from the participants' design and reflection processes. Following that, we discuss two pedagogical insights that were reflected in the participants' written descriptions of their work with their students.

Mathematical insight #1: The importance of differentiating between local and global interpretations of graphs. One of the most widely reported insights by participants, most notably and explicitly at Site B, was that their students' activity in the software environment provided occasions to gauge whether the students were making local or global interpretations of the features of a graph. This focus was discussed at Site B after the participants had read Monk's (1192) distinction between local interpretations, which involve attention to specific point-wise features such as points of intersection or relative extrema, and global interpretations, which involve a focus on the behavior of the graph over its entire domain. For example, one participant reported that his students, who had investigated periodic graphs, were easily able to interpret the features of both the position and the velocity graphs as the character in his story walked back and forth between the garage and the end of the driveway. The participant described how his student, Jake, interpreted the situation:

> When identifying the period of the function, Jake focused on the maximums and minimums of the position graph. Jake has given meaning to those features as ones that will help him find the period of the function. After the period was identified on the position graph, Jake would verify the period on the velocity graph. Jake has identified the maximums and minimums of the position graph as places where Toni [the character in the story] is at the end of the driveway or at the garage. When he is identifying the period of functions, he is immediately drawn to these features even if the initial position is not a relative extrema of the graph.

Although this participant remained skeptical about Jake's understanding of the relationship between the direction the character was moving and the slope of the position graph, his final report indicated that he felt that his pupil had developed a good understanding of the relationship between the relative extrema (a local feature of the graph) and the period of the function as well as a solid interpretation of these features in terms of the character's actual motion. This distinction, which the participant made on his own, is significant in that it reflected his shift away from a focus on correct or incorrect answer toward mathematical meaning in terms of interpretation. It also demonstrated strong pedagogical content knowledge in that he distinguished between his student's formal understandings of the mathematical concepts (such as slope as an interpretation of speed) and his student's interpretation of the various critical points on the graph.

Another participant, Romy, explicitly focused her lesson on her student's interpretation of across-time or global features of a function. Romy wanted her students to predict the total distance a character would travel given a velocity graph and to create a position graph that would match one character's motion traveling at two different speeds. To Romy's surprise however, one of her students created a graph resembling that shown in Figure 4, which is a non-standard representation that led Romy to re-think how graphs may be interpreted in unanticipated ways. Romy expected her students to use piecewise constant line segments to represent the character's changing velocity over time. This interpretation would have involved a global interpretation of a position graph. In assessing her student's work, Romy allowed for the possibility that the student may have misunderstood the question but she also felt confident that the student understood the sequential nature of the motion. What is more significant is that this participant appeared to be flexible enough in her own mathema-

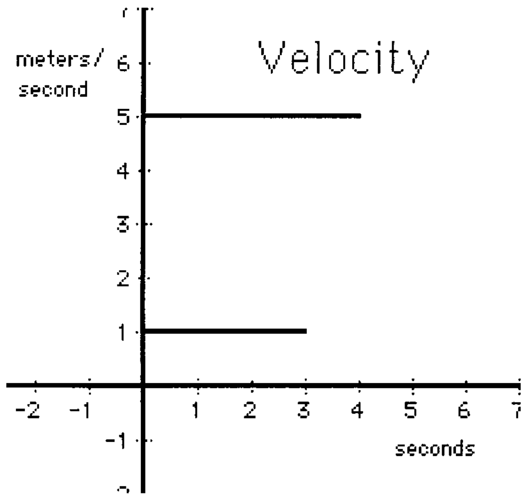


Figure 4. An unexpected velocity graph generated by Romy's student.

tical thinking to extend her mathematical interpretation of the graphical representation by seeing how the student could be reading along only one dimension of the graph and not making time an explicit part of the representation. By considering the student's paper and pencil representation as an alternative way of knowing, the participant gained insight into how and why the student might choose an unconventional representation of sequential, one-dimensional motion.

Mathematical insight #2: The importance of appropriate contexts. All participants created contexts that they thought would appeal to their students' interests, but several reported that this was more difficult than they had anticipated. For example, one participant from Site A explained that he began his sequence using the context of plant growth. Later he realized that this caused a problem when he wanted to include negative velocity into the same context.

The context of the overtake race, where the one-dimensional motion of one character overtakes another character given a range of initial starting conditions and varying velocities, as featured in the Cheetah and Gazelle sequence, became a powerful metaphor for supporting the participants' development of short instructional sequences. All but one of the participants at Site B and approximately half of the participants at Site A used some variation of a race in teaching their mathematical content.

For example, one participant who designed an activity to focus on Riemann Sums used the context of an overtake race to feature one dot's velocity controlled by a half period of a sine curve. The task for his students was to create another velocity graph using piecewise, horizontal line segments so that the second dot would end up in a tie with the first dot. This participant drew on the students' prior knowledge about the area under the velocity curve as determining position and created a mathematical environment that would allow his students to create a piecewise approximation for the area under a sine curve. Based on his two students' work, in which they generated their approximations by positioning the horizontal line segments at the midpoint of the curve, this participant refined his activity to include both upper and lower sums for the approximation. What we find significant is that this refinement occurred as a result of seeing his students interact with each other and with the learning task that he designed. For this participant, the mathematical insight that he experienced emerged as he tried to reconcile the contextual difference for his students between approximating by positioning segments at the midpoint of the curve on an interval and by sandwiching the curve between upper and lower segments.

Pedagogical Insights from Participants as Teachers

Pedagogical insight #1: The value of building on students' incorrect explanations. Although some of the participants such as Romy capitalized on her student's errors, other participants from both sites maintained limited views of what constituted a correct answer from a pedagogical point of view. We see this as a stumbling block when it prevents participants from building on their students' incorrect but reasonable and potentially fruitful explanations.

One participant who missed an opportunity to capitalize on her student's own mathematical explanations was Ellen, a participant from Site A who was tutoring a talented high school sophomore who had never seen a position or velocity graph. To begin Ellen asked him to draw a position graph in which a clown was traveling at a constant rate of 6 m/s for 3 seconds. The student reasoned that the clown would be at 18 meters after three seconds. He then plotted the points (3,18) and (0,0) and then drew a line segment connecting these two points as shown in Figure 5a. Ellen then asked him to draw a velocity vs. time graph showing the clown's velocity at each second. In response, the student drew the graph shown in Figure 5b. Because this graph did not match Ellen's expectations of what the student would draw, namely a velocity graph showing a constant speed of 6 m/s for 3 seconds, she dismissed it as incorrect.

However, in her written reflection, Ellen realized that her student had described a graph of rate by *seeing* velocity in the ratios of the height and length of each stair step. She noted that her student explained that

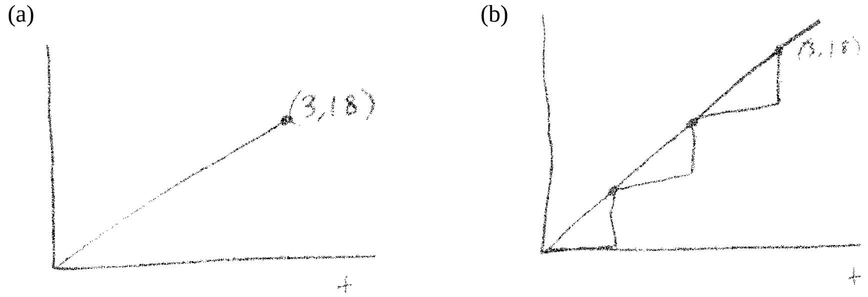


Figure 5. (a) Student's drawing of position graph. (b) Student's drawing of velocity graph.

if the person was traveling for 6 meters per second, then, after 1 second, he would be at the point (1,6), which could be calculated as a speed of 6 meters per second. Likewise, after two seconds, he would be at the point (2,12), which, the student explained, was 12 meters per two seconds, or 6 meters per one second. For him, although they looked the same, graphs 5a and 5b were entirely different, depending on how he interpreted them. After discussing this with her instructor, Ellen gained insight into the potential value of building on students' intuitions rather than assuming that she could just erase what he was thinking and tell him how to create the correct velocity graph.

A second example of this tension was evident in one participant's report from Site B. Linda described a student's desire to work with the simulation to check his conjectures and contrasted that with her drive toward interpreting it graphically and then with the formula for area. Even when the formalism followed the exploration in the lesson plan, Linda maintained that the formalism was her central goal for the lesson. She essentially saw the lesson as "failing" (her words) because, in the end, the student did not move successfully, in her opinion, to the formalism. This report, like several others, indicated that the participants' experiences as students and teachers highlighted the pedagogical dilemma of the relationship between the explorations of ideas and their expression in the formalism of mathematical symbols as described in the earlier section on pedagogical insights from participants as students.

Pedagogical insight #2: Influence of hidden supports and constraints of technology on students' mathematical activities. As Greeno (1997) and Cobb (1999) pointed out, the features of any computer software program profoundly affect the nature of one's activity with it. One implication of this claim is that teachers wishing to use technology in their classrooms need to recognize the supports and constraints of the technology and to decide how to structure activities so that students act in ways that are potentially productive. For example, one technological feature that may have constrained the types of mathematical activity in which the students were engaged was indicated in several participants' descriptions of their students' dissonance with discontinuous velocity graphs. Julie, a participant from Site B, who worked with a group of four 11 year-old students, reported the following dialogue about the velocity graph shown in Figure 6.

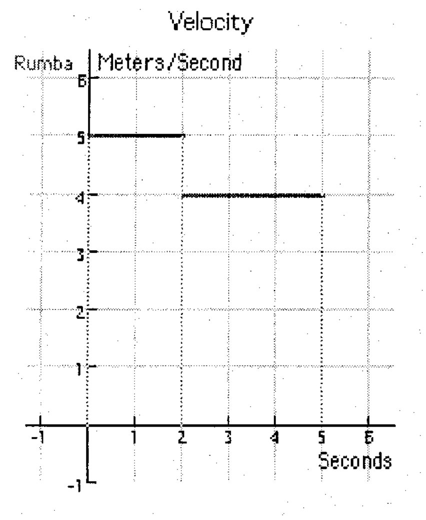


Figure 6. A discontinuous velocity graph.

A: But that's impossible. Rumba [a clown in the Mathworlds software] has to pass through all the velocities between 5 m/s and 4 m/s before actually reaching 4 m/s.
T: Why do you say that?
B: That means Rumba would have to be an alien!
C: Yea, there has to be some number of seconds when Rumba is slowing down before reaching the new slower velocity.

Julie interpreted the dialogue as follows:

The limited ability of MathWorlds to replicate real-world situations became apparent to even the sixth graders with whom I worked. In fact, they saw what perhaps the creators of MathWorlds failed to see – the physical impossibility of instantaneously going from walking 5 meters/second to walking 4 meters/second. . . . Hence, in the eyes of these students, discontinuous velocity graphs seemed as extraterrestrial as discontinuous position graphs.

Julie's comment indicates that she understood that the use of discontinuous velocity graphs was an artifact of the design of the software but saw this as a problematic mismatch with the viable explanation the student had created to explain his experienced reality. For Julie, the designers' decision was not consistent with her intended pedagogy. She therefore interpreted the constraint as presenting a serious pedagogical difficulty in that it pushed the students to abandon their common sense about realistic situations.

The participants at Site A discussed the differences between the affordances of the MathWorlds software and those of the motion detector. In their discussion, they noted that because the motion detector did reflect the real-world, it contained noise that distracted from the mathematical abstraction of the motion. Their work in MathWorlds following their work with the motion detector enabled them to differentiate the mathematical abstraction from the noise and to develop a deeper understanding of the way in which the limits of real-valued functions serve as a bridge between motion in the real world and that of mathematical formalisms that model it.

DISCUSSION

The two research questions that we set out to investigate were: (a) How do participants, acting in the role of *student*, think about the mathematics of change when using an exploratory microworld, and (b) How do participants, acting in the role of *teacher*, plan, implement, and reflect on lessons about the mathematics of change when tutoring their students. In analyzing our results, we found it useful to coordinate the participants' experiences in our classes with the perspectives through which they enacted those experiences. In so doing, we were looking for a way to view our courses from students' experiential perspectives, and perhaps gain more insight than just looking at their pedagogical thinking as prospective teachers (Cell IV) or their mathematical knowledge as students (Cell I).

Our analysis revealed two critical perspectives that might have been otherwise missed. First, the participants developed pedagogical insights as students of mathematics (Cell II), and second, the participants developed mathematical insights as teachers of mathematics (Cell III). Some of the participants' most powerful pedagogical insights emerged as they were assuming the role of mathematics students. For example, the debates over the value of conceptual explanations and the benefits of exploring mathematical ideas (with the use of microworlds) before introducing formalizations were rich because the participants argued from both perspectives. Although neither debate was fully resolved, the participants from both sites came to value their own mathematical insights and appreciate these values as teachers more deeply than had they just been told of these pedagogical strategies in a methods class. In other words, we found that the students who experienced the struggles from both sides came to develop an appreciation for the value of conceptual explanations and explorations with technology.

A second finding that confirmed our hypothesis regarding the value of viewing participants in the dual roles was that some of the participants' mathematical insights developed as they created, taught, and reflected on mathematical lessons. For example, many participants realized the importance and difficulty of choosing a rich context through which one could explain the mathematics underlying the relationship between velocity graphs and their associated position graphs. Similarly, several participants from each site found that they could learn new mathematics by listening to their students' interpretations.

The constructivist approach we assumed when planning and analyzing this project focused on accounting for cognitive changes which were situated in the context in which the individuals were acting. In following Steffe and Thompson's (2000) premise that the source of perturbations is often the social situation in which the student is acting, we are not claiming that the microworld alone caused any of these fruitful reorganizations. Instead, we claim that as the activities were realized in the social setting of each of two sites, the participants' efforts to reconcile what they anticipated with what they found led to fruitful discussions. It was these discussions and the participants' consequent reflections and abstractions that we believe led to changes in their mathematical and pedagogical content knowledge as well.

CONCLUSIONS

We close with two conclusions that relate to our work as teacher educators. First, as noted above, we found that our use of computer-based activity sequences served as an effective means for eliciting perturbations among prospective and practicing teachers. Our results indicate that many of the participants experienced "Aha" insights because they reorganized their initial understandings of the mathematical relations between position and velocity, and, in so doing, gained a deeper understanding of the mathematical formalizations as well. We found that because the activities were somewhat novel, they placed the participants in a learning situation in which familiar mathematics seemed unfamiliar but not threatening, thus evoking a dissonance that needed to be resolved.

A second implication for teacher educators is that our analytic framework enabled us to explicitly position our participants in the dual roles of student and teacher while simultaneously considering their mathematical and pedagogical knowledge. The value of this conceptualization can be illustrated by considering a discussion that took place near the beginning of the semester at Site A. When the instructor asked the participants what the value of a conceptual explanation was, they seemed to almost uniformly agree that good teaching involved "delivering clear explanations" to their own students and hence the value of a conceptual explanation was that it enabled them to explain things more clearly. This view changed over the course of the semester, such that they came to value conceptual explanations not as tools for preaching but as tools for helping their own students explain things for themselves. Had this not been an explicit discussion early on, neither the participants nor the instructor at Site A would have been aware of the fact that they were talking past each other.

NOTES

[1] The analysis reported in this paper was supported in part by the National Science Foundation under grant No. REC-9619102. The opinions expressed do not necessarily reflect the views of the Foundation.

[2] The version of MathWorlds software used in our classes was compatible with the Macintosh platform only. A Java version is now available at http://www.simcalc.umassd.edu/.

REFERENCES

Bowers, J. S. & Nickerson, S. D. (in press). Students' changing views of rates and graphs when working with a simulation microworld. *Focus on Learning Problems in Mathematics*.

Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning*, 1, 5–43.

Cooney, T. J. & Wilson, M. R. (1993). Participants' thinking about functions: Historical and research perspectives. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (131–158). Hillsdale, NJ: Erlbaum.

Cooney, T. J., Wilson, M. R., Albright, M. & Chauvot, J. (1998, April). *Conceptualizing the professional development of secondary preservice mathematics teachers*. Paper presented at the annual conference of the American Educational Research Association, San Diego, CA.

Davis, R. B. & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, 5, 281–303.

Doerr, H. M. (1997). Experiment, simulation, and analysis: An integrated instructional approach to the concept of force. *International Journal of Science Education*, 19, 265–282.

Greeno, J. G. (1997). On claims that answer the wrong questions. *Educational Researcher*, 26(1), 5–17.

Harel, G., Behr, M., Lesh, R. & Post, T. (1994). Invariance of ratio: The case of children's anticipatory scheme for constancy of taste. *Journal for Research in Mathematics Education*, 25, 324–345.

Hiebert, J. (Ed.) (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Erlbaum.

Hiebert, J. & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (65–97). New York: Macmillan.

Kaput, J. (1994). The representational roles of technology in connecting mathematics with authentic experience. In R. Biehler, R. W. Scholz, R. Sträber & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (189–199). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Kaput, J. & Roschelle, J. (1997). Deepening the impact of technology beyond assistance with traditional formalism in order to democratize access to ideas underlying calculus. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (105–112). Lahti, Finland: University of Helsinki.

Lampert, M. & Ball, D. (1998). *Teaching, multimedia, and mathematics: Investigations of real practice*. New York: Teachers College Press.

Lauten, A. D., Graham, K. J. & Ferrini-Mundy, J. (1994). Student understanding of basic calculus concepts: Interaction with the graphics calculator. *Journal of Mathematical Behavior*, 13, 225–237.

Lee, L. & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics*, 20, 41–54.

Leinhardt, G., Zaslavsky, O. & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, 60, 1–64.

Lobato, J. & Thanheiser, E. (1999). Re-thinking slope from quantitative and phenomenological perspectives. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, 291–297). East Lansing, MI: ERIC Clearinghouse for Sciences, Mathematics, and Environmental Education.

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (175–193). Washington, DC: Mathematical Association of America.

Nemirovsky, R. & Monk, S. (2000). "If you look at it the other way . . .": An exploration into the nature of symbolizing. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (177–221). Mahwah, NJ: Erlbaum.

Pajares, F. & Bengston, J. K. (1995, April). *The psychologizing of teacher education: Formalist thinking and preservice teachers' beliefs*. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (41–52). Reston, VA: National Council of Teachers of Mathematics.

Schoenfeld, A. H., Dubinsky, E. L. & Gleason, A. (Eds.). (1997). *MAA notes: No. 43. Student assessment in calculus: A report of the NSF working group on assessment in calculus*. Washington, DC: Mathematical Association of America.

Selden, J., Mason, A. & Selden, A. (1989). Can average calculus students solve nonroutine problems? *Journal of Mathematical Behavior*, 8, 45–50.

Siebert, D., Lobato, J. & Brown, S. (1998). Understanding how prospective secondary teachers avoid accommodating their existing belief systems. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood & L.Stiff (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group*

for the Psychology of Mathematics Education (Vol. 2, 620–626). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.
Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher*, 26(3), 9–15.
Steffe, L. P. & Thompson, P. W. (2000). Interaction or intersubjectivity? A reply to Lerman. *Journal for Research in Mathematics Education*, 31, 191–209.
Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (495–511). New York: Macmillan.
Thompson, A. G. & Thompson, P. W. (1996). Talking about rates conceptually, part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, 27, 2–24.
Thompson, P. W. (1992). Notations, conventions, and constraints: Contributions to effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education*, 23, 123–147.
Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 26, 229–274.
Thompson, P. W. (1996). Imagery and the development of mathematical reasoning. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (267–285). Mahwah, NJ: Erlbaum.
von Glasersfeld, E. (1987). Learning as a constructive activity. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (3–17). Hillsdale, NJ: Erlbaum.
Von Glasersfeld, E. (1995). A constructivist approach to teaching. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (3–15). Hillsdale, NJ: Erlbaum.
Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, 22, 219–236.
Wilson, M. & Goldenberg, M. P. (1998). Some conceptions are difficult to change: One middle school mathematics participant's struggle. *Journal of Mathematics Teacher Education*, 1, 269–293.

Janet Bowers
Center for Research in Mathematics and Science Education
College of Sciences
San Diego State University
San Diego, CA 92120
USA
E-mail: jbowers@sunstroke.sdsu.edu

Helen M. Doerr
Department of Mathematics
215 Carnegie Library
Syracuse University
Syracuse, NY 13244-1150
USA
E-mail: hmdoerr@sued.syr.edu